Vanishing of cohomology

Theorem 1 *Let $\mathcal{A}$ be an abelian category with enough injectives and let $\Gamma\colon \mathcal{A} \to \mathcal{B}$ be a left exact additive functor. Suppose that $\mathcal{F}$ is a full subcategory of $\mathcal{A}$ with the following properties:*

1. *Whenever $0 \to F' \to F \to F'' \to 0$ is an exact sequence in $\mathcal{A}$, if F' and F belong to $\mathcal{F}$,then so does F'', and the sequence $0 \to \Gamma(F') \to \Gamma(F) \to \Gamma(F'') \to 0$ is exact.*

2. *Every object of $\mathcal{F}$ can be embedded in an object of $\mathcal{F}$ which is injective as an object of $\mathcal{A}$.*

Then $R\Gamma^i(F) = 0$ for every $F \in \mathcal{F}$ and every $i > 0$.

Proof: If F is an object of $\mathcal{F}$, then by hypothesis we can find an embedding $F \to F^0$ where $F \in \mathcal{F}$ and F is injective in $\mathcal{A}$. Then the quotient $Q^1 =: F^0/F$ still belongs to $\mathcal{F}$, and hence can be embedded in an object F^1 of $\mathcal{F}$ which is injective in $\mathcal{A}$. Continuing in this way, we find a resolution $\epsilon\colon F \to F^\cdot$ such that each F^q belongs to $\mathcal{F}$, is injective in $\mathcal{A}$, and furthermore such that $Z^q =: Ker(d^q)$ belongs to $\mathcal{F}$ for every q. For each q we have an exact sequence:

$$0 \to Z^q \to F^q \to Z^{q+1} \to 0,$$

and since all the terms lie in $\mathcal{F}$, the sequence remains exact after we apply the functor Γ. Thus $R^q\Gamma(F^\cdot) = H^q(\Gamma(F^\cdot) = 0$ for $q > 0$. □

Corollary 2 *If X is a topological space and F is a flasque abelian sheaf on X, then $H^q(X, F) = 0$ for $q > 0$.*

Proof: First we prove that any F-torsor T on X is trivial. Using the Hausdorff maximality principle, one reduces easily to proving that if X can be covered by two open sets U_1 and U_2 such that $T(U_1)$ and $T(U_2)$ are nonempty, then $T(X)$ is nonempty. Choose $t_i \in T(U_i)$ and let $f \in F(U_1 \cap U_2)$ be the unique element such that $ft_{1|_{U_1\cap U_2}} = t_{2|_{U_1\cap U_2}}$. Since F is flasque, we can choose an $f' \in T(U_1)$ extending f, and then setting $t_1' = f't_1$, we find that t_1' and t_2' patch.

It follows now that if $0 \to F' \to F \to F'' \to 0$ is an exact sequence in Ab_X with F' and F flasque, then for every open set U of X, we find a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F'(X) & \longrightarrow & F(X) & \longrightarrow & F''(X) & \longrightarrow & 0 \\
 & & \downarrow{\rho'} & & \downarrow{\rho} & & \downarrow{\rho''} & & \\
0 & \longrightarrow & F'(U) & \longrightarrow & F(U) & \longrightarrow & F''(U) & \longrightarrow & 0
\end{array}$$

In this diagram ρ is surjective because F is flasque and it follows that ρ'' is surjective. Thus F'' is flasque, and the category of flasque sheaves satisfies (1.1). Since every injective is flasque, it also satisfies (1.2). □

Theorem 3 *Suppose that X is a topological space and $\mathcal{B}$ is a base for its topology which is closed under finite intersection and such that each $U \in \mathcal{B}$ is quasi-compact. Let F be a sheaf of abelian groups on X. Then the following are equivalent:*

1. *For every $U \in \mathcal{B}$, $H^q(U, F) = 0$ for $q > 0$.*
2. *For every finite open cover $\mathcal{U} \subseteq \mathcal{B}$ of an element U of $\mathcal{B}$, the Cech cohomology $\check{H}^q(\mathcal{U}, F)$ of F with respect to $\mathcal{U}$ vanishes.*

Proof: We omit the proof that (1) implies (2). To prove that (2) implies (1), consider the set $\mathcal{A}$ of all abelian sheaves on X satisfying (2). We claim that if $F \in \mathcal{A}$ then $H^q(U, F) = 0$ for $q > 0$ and $U \in \mathcal{B}$. Without loss of generality, we may assume that $X \in \mathcal{B}$, and it will suffice to prove that $H^q(X, F) = 0$ for $q > 0$. By Hartshorne (II 4.3), $\mathcal{A}$ contains all flasque sheaves, and in particular all injective sheaves. So by (1), it will suffice to prove that $\mathcal{A}$ satisfies (1.1). Suppose that $0 \to A \to B \to C \to 0$ is an exact sequence of abelian sheaves on X and A and B belong to $\mathcal{A}$. If $\mathcal{U} \subseteq \mathcal{B}$ is a finite open cover of an element U of $\mathcal{B}$, then by hypothesis the Cech cohomology groups $\check{H}^q(\mathcal{U}, A)$ vanish for $q > 0$, and in particular for $q = 1$. Since any $U \in \mathcal{B}$ is quasi-compact, it follows that any A-torsor on U is trivial, and hence that the sequence

$$0 \to A(U) \to B(U) \to C(U) \to 0$$

is exact. Now if $\mathcal{U}$ is any finite subset of $\mathcal{B}$, it follows that for any multi-index I and any $I \to \mathcal{U}$, the intersection U_I belongs to $\mathcal{B}$, and hence the sequence

$$0 \to \prod_I A(U_I) \to \prod_I B(U_I) \to \prod_I C(U_I) \to 0$$

is exact. In other words, we get an exact sequence of complexes:

$$0 \to \check{C}^{\cdot}(\mathcal{U}, A) \to \check{C}^{\cdot}(\mathcal{U}, B) \to \check{C}^{\cdot}(\mathcal{U}, C) \to 0.$$

Taking the long exact sequence of cohomology we find the exact seqeunce

$$\check{H}^q(\mathcal{U}, B) \to \check{H}^q(\mathcal{U}, C) \to \check{H}^{q+1}(\mathcal{U}, A).$$

Since A and B belong to $\mathcal{A}$, we deduce that $\check{H}^q(\mathcal{U}, C) = 0$ for $q > 0$ if $\mathcal{U}$ is a cover of an element of $\mathcal{B}$. □

Theorem 4 *If X is an affine scheme and F is a quasicoherent sheaf on X, then $H^q(X, F) = 0$ for $q > 0$.*

Proof: Thanks to the previous result, it will suffice to show that if $\mathcal{B}$ is the set of special affine open subsets of X and $\mathcal{U}$ is any finite cover of an element U of $\mathcal{B}$, then the Cech cohomology $\check{H}^q(U, F) = 0$ for $q > 0$. Note first that if $j: U \to X$ is the inclusion map, then $j_* j^* F$ is quasicoherent on X, because the j is a quasicompact and quasiseparated map. The same applies to the inclusion of any U_I, and since $\mathcal{U}$ is finite, we see that all the terms of the "sheaf" Cech complex $\underline{C}^{\cdot}(\mathcal{U}, F)$ are quasicoherent. This complex thus defines a resolution of F by quasicoherent sheaves, and since the global section functor is exact on the category of quasicoherent sheaves, the complex remains exact when we apply Γ. Thus the global Cech complex is acyclic, and the result is proved. □